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An Extension of the Theory of Orthogonal Polynomials and Gaussian Quadrature to Trigonometric and Hyperbolic Polynomials

PER ERIK KOCH*

Institute of Informatics, University of Oslo, Oslo, Norway

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A class of functions was introduced by P. E. Koch (Research Report No. 72, Institute of Informatics, University of Oslo, 1982) that includes the algebraic, trigonometric, and hyperbolic polynomials, but retains sufficient similarities to the algebraic polynomials so that it was possible to generalize the Jackson theorems. This class need not satisfy the Haar condition. Some of the properties of orthogonal polynomials are extended here to this class. In particular, the recurrence relation, the Erdös–Turan theorem, the Dini condition for pointwise convergence of orthogonal expansions, and some of the basic properties of Gaussian quadrature are generalized.

1. INTRODUCTION

In [5] the author introduced a class of functions that includes the algebraic, trigonometric, and hyperbolic polynomials, but retains sufficient similarities to the algebraic polynomials so that it was possible to generalize the Jackson theorems. In this paper we extend some of the properties of orthogonal polynomials to this class. In particular, we generalize the recurrence relation (Section 2), the Erdös–Turan theorem (Section 3), the Dini condition for pointwise convergence of orthogonal expansions (Section 4), and some of the basic properties of Gaussian quadrature (Section 5).

Before we go any further, let us identify this class.

DEFINITION 1.1. Let $I=[a,b]$ and $\mu=(a+b)/2$. For functions $f\colon I\to\mathbb{R}$ define $\check{f}\colon I\to\mathbb{R}$ by $\check{f}(x)=f(2\mu-x)$, $x\in I$. Call f even if $\check{f}=f$, odd if $\check{f}=-f$. Let s, c_1, c_2 be continuous functions that are different from zero almost everywhere, where s is odd and c_1, c_2 are even. Put

$$V_n=\operatorname{span}\{c_1, c_2 s, c_1 s^2, c_2 s^3, ..., f_k, ..., f_n\} \tag{1.1}$$

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where the kth basis function is

$$f_k = s^{k-1} \begin{cases} c_1, & k \text{ odd}, \\ c_2, & k \text{ even}, \end{cases} \qquad k = 1, 2, \ldots . \tag{1.2}$$

We notice the following connection between V_n and the polynomials $P_n = \operatorname{span}\{1, x, \ldots, x^{n-1}\}$: every $\phi \in V_n$ may be written in the form

$$\phi = c_1(p \circ s) + c_2(q \circ s) \tag{1.3}$$

where $p \in P_n$ is even, $q \in P_n$ is odd, and $\circ$ denotes the composite function operator. Especially, if $\phi \in V_n$ is even, then

$$\phi = c_1(p \circ s), \qquad p \in P_n, p \text{ even}, \tag{1.4}$$

and if $\phi \in V_n$ is odd, then

$$\phi = c_2(q \circ s), \qquad q \in P_n, q \text{ odd}. \tag{1.5}$$

Also

$$\phi \in V_n \Rightarrow s^2 \phi \in V_{n+2}, \qquad n \geqslant 1. \tag{1.6}$$

In fact, we get $V_n = P_n$ in the case $s(x) = x - \mu$, $c_1(x) = c_2(x) = 1$, $x \in I$.

V_{2m+1} becomes the trigonometric polynomials of degree $\leqslant m$ if we use $s(x) = \sin((x-\mu)/2)$, $c_1(x) = 1$, and $c_2(x) = \cos((x-\mu)/2)$, $x \in I$. In this case (1.1) will also define even dimensional trigonometric spaces lying between these odd dimensional ones. The interval I may be any closed one with length $\leqslant 2\pi$.

By choosing $s(x) = \sinh((x-\mu)/2)$, $c_1(x) = 1$, $c_2(x) = \cosh((x-\mu)/2)$ we obtain hyperbolic polynomials (see Section 3 for definitions).

In general, V_n need not satisfy the Haar condition on I. But if the interpolation points lie symmetrically around the midpoint μ the interpolation problem will have a unique solution (see Section 3). Also Gauss quadrature rules exist when the weight function is even (Section 5).

2. A Recurrence Relation

Let us first give some notation. Define the inner product $(\ ,\ )$ by

$$(f, g) = \int_a^b f(x)\, g(x)\, w(x)\, dx \tag{2.1}$$

where w is Riemann integrable and positive. Let $L_w^2(I)$ be the underlying Hilbert space and $\|\ \|$ the induced norm.

We will use the Gram–Schmidt process to orthogonalize the sequence $f_1, f_2, f_3, \ldots$ given by (1.2). First we must show that $f_1, \ldots, f_n$ are linearly independent for each n. Suppose not. Then by (1.1) and (1.3) there are polynomials $p, q \in P_n$, not both of them identically zero, such that $c_1(p \circ s) + c_2(q \circ s) = 0$, where p is even and q is odd. Set $\phi = c_1(p \circ s) + c_2(q \circ s)$. Then $c_1(p \circ s) = \frac{1}{2}(\phi + \check{\phi}) = 0$. The polynomial p must vanish on a set of positive measure. Hence $p = 0$. Similarly $q = 0$. This contradiction implies the linear independence. Now we may use the Gram–Schmidt process. Put

$$\phi_n = f_n - \sum_{k=1}^{n-1} \frac{(f_n, \phi_k)}{(\phi_k, \phi_k)} \phi_k, \qquad n = 1, 2, \ldots . \tag{2.2}$$

$\{\phi_1, \ldots, \phi_n\}$ is then an orthogonal basis for V_n. $\{\bar{\phi}_1, \ldots, \bar{\phi}_n\}$ is an orthonormal basis for V_n, where $\bar{\phi}_k = \phi_k / \|\phi_k\|$.

Notice that $f_1, f_3, f_5, \ldots$ are even and that $f_2, f_4, f_6, \ldots$ are odd. If the weight function w is even then by (2.2), $\phi_1 = f_1$ is even, $\phi_2 = f_2$ is odd, $\phi_3 = f_3 - ((f_3, \phi_1)/(\phi_1, \phi_1))\, \phi_1$ is even, and so on. Using induction on n we obtain by (2.2)

$$\check{\phi}_n = (-1)^{n-1} \phi_n, \qquad n = 1, 2, \ldots . \tag{2.3}$$

These orthogonal functions possess a five term recurrence relation. This result generalizes that of [11].

THEOREM 2.1. *Let $f_1, f_2, f_3, \ldots$ be as in Definition* 1.1. *The orthogonalized sequence given by* (2.2) *possesses the following recurrence relation*:

$$\begin{aligned} \phi_{n+1} &= -\alpha_{n,n}\phi_n + (s^2 - \alpha_{n,n-1})\, \phi_{n-1} - \alpha_{n,n-2}\phi_{n-2} - \alpha_{n,n-3}\phi_{n-3} \\ \alpha_{n,j} &= (s^2\phi_{n-1}, \phi_j)/\|\phi_j\|^2, \qquad j = n-3, \ldots, n, \ n \geqslant 4. \end{aligned} \tag{2.4}$$

If w is even then $\alpha_{n,n} = \alpha_{n,n-2} = 0$.

Proof. By (1.6), $s^2\phi_k \in V_{k+2}$. By (2.2) there are $a_{n,j} \in \mathbb{R}$ such that

$$\phi_{n+1} - s^2\phi_{n-1} = -\sum_{j=1}^{n} a_{n,j}\phi_j.$$

Now proceed as in the proof of the recurrence relation for orthogonal polynomials. Taking inner products with ϕ_j yields $a_{n,j} = \alpha_{n,j}$, $j = n-3, \ldots, n$, and $a_{n,j} = 0$, $j < n-3$. If w is even then by (2.3) $s^2\phi_{n-1}\phi_n w$ and $s^2\phi_{n-1}\phi_{n-2} w$ are odd. So $\alpha_{n,n} = \alpha_{n,n-2} = 0$. ∎

If we set $\phi_0 = \phi_{-1} = 0$, then (2.4) is valid for $n \geqslant 2$.

Since $s^2\phi_{n-3}$ may be written in the form $\phi_{n-1}+\sum_1^n \gamma_j\phi_j$ we find that also $\alpha_{n,n-3}=\|\phi_{n-1}\|^2/\|\phi_{n-3}\|^2$.

If $c_1=c_2$ then (2.4) may be replaced by a three term recurrence relation (since $\phi\in V_k \Rightarrow s\phi\in V_{k+1}, k\geqslant 1$).

$$\phi_{n+1}=(s-\alpha^1_{n,n})\,\phi_n-\alpha^1_{n,n-1}\phi_{n-1},\ \alpha^1_{n,j}=(s\phi_n,\phi_j)/\|\phi_j\|^2.$$

In particular, the recurrence relation for orthogonal polynomials is obtained in the case $c_1=c_2=1$, $s(x)=x-\mu$.

For the orthonormal sequence we have ·

$$\beta_{n,n+1}\bar{\phi}_{n+1}=-\beta_{n,n}\bar{\phi}_n+(s^2-\beta_{n,n-1})\,\bar{\phi}_{n-1}-\beta_{n,n-2}\bar{\phi}_{n-2}-\beta_{n,n-3}\bar{\phi}_{n-3}$$
$$\beta_{n,j}=(s^2\bar{\phi}_{n-1},\bar{\phi}_j),\qquad j=n-3,\dots,n+1,\quad n\geqslant 4. \tag{2.5}$$

3. Interpolation

We will first show that under certain mild conditions on c_1, c_2, s, interpolation from V_n is possible when the interpolation points lie symmetrically around the midpoint of the interval.

LEMMA 3.1. *Assume that s is strictly monotone on I. Let $n\in\mathbb{N}$ and let $\xi_1,\dots,\xi_n$ be n distinct points in I, located symmetrically around $\mu=(a+b)/2$,*

$$\xi_{n+1-j}=2\mu-\xi_j,\qquad j=1,\dots,n. \tag{3.1}$$

Suppose that $c_1(\xi_j)\neq 0$, $c_2(\xi_j)\neq 0$, $j=1,\dots,n$. Then for any $f\in C(I)$ there is a unique function ϕ in V_n that solves the interpolation problem

$$\phi(\xi_j)=f(\xi_j),\qquad j=1,\dots,n. \tag{3.2}$$

Proof. Any $\phi\in V_n$ may be written in the form (1.3). Since c_1, c_2, p are even and s and q are odd and (3.1) holds, (3.2) is equivalent to

$$\left.\begin{aligned} c_1(\xi_j)\,p(s(\xi_j))&=\tfrac{1}{2}(f(\xi_j)+\check{f}(\xi_j)),\\ c_2(\xi_j)\,q(s(\xi_j))&=\tfrac{1}{2}(f(\xi_j)-\check{f}(\xi_j)),\end{aligned}\right.\qquad j=1,\dots,n. \tag{3.3}$$

This uniquely determines p and q. Since $\frac{1}{2}(f+\check{f})/c_1$ $(\frac{1}{2}(f-\check{f})/c_2)$ is even (odd), p is even and q is odd. ∎

The following conditions on c_1, c_2, s will often be sufficient for our purposes:

ASSUMPTION 3.2. Let $I=[a,b]$. Suppose that $c_1, c_2\in C(I)$ are even and that $c_1(x) > 0$, $c_2(x) > 0$, $a < x < b$. Suppose also that $s \in C^1(I)$ is odd, $s'(x) > 0$, $a < x < b$, and that $(w \circ s^{-1}) \cdot (s^{-1})'$ is Riemann integrable, where w is the weight function.

From the classical Erdös–Turan theorem we know that it is advantageous to use the zeroes of orthogonal polynomials as interpolation points. To extend this theorem we must first verify that ϕ_{k+1} has k zeroes in I. The following proposition generalizes a result in [12, p. 245].

PROPOSITION 3.3. *Assume that c_1, c_2, s satisfy Assumption 3.2 and that $c_i(x) \neq 0$, $x = a, b$, $i = 1, 2$. Suppose further that the weight function w is even. Then for each $k \in \mathbb{N}$, ϕ_k has exactly $k-1$ zeroes on I. They are all simple, contained in (a, b), and they are located symmetrically around μ.*

Proof. Equations (2.3), (1.4), and (1.5) imply

$$\begin{aligned} \phi_k &= c_1(p_k \circ s), \qquad && k \text{ odd} \\ &= c_2(q_k \circ s), && k \text{ even} \end{aligned}$$

where p_k is even and q_k is odd. First let k be odd. A simple change of variable shows that $p_1, p_3, p_5, \ldots$ are orthogonal with respect to the weight function $v_1 = (c_1 \circ s^{-1})^2 (w \circ s^{-1})(s^{-1})'$. $p_1, p_3, p_5, \ldots$ are all even. Since v_1 is even, $p_1, p_3, p_5, \ldots$ may be extended to a full sequence of polynomials, $p_1, p_2, p_3, \ldots$, orthogonal with respect to v_1. This is because $p_2, p_4, p_6, \ldots$ would be all odd anyway and independent of $p_1, p_3, p_5, \ldots$. From the classical theory we know that p_k has exactly $k-1$ zeroes and that these are all simple and lying in $(s(a), s(b))$. Since v_1 is even, they also lie symmetrically around 0. Since $\phi_k = c_1(p_k \circ s)$, this holds for ϕ_k too. The case of k even is proved in exactly the same way. ∎

We apply Proposition 3.3 in the special case that $c_1 = c_2 = 1$. Let us use the notation

$$U_n = \operatorname{span}\{1, s, \ldots, s^{n-1}\}, \qquad n \geqslant 1. \tag{3.4}$$

We now come to the extended Erdös–Turan theorem.

THEOREM 3.4. *Suppose that c_1, c_2, s satisfy Assumption 3.2. Let w be an even weight function. Let $\| \ \|$ be the norm induced by w. Let $\psi_1, \psi_2, \psi_3, \ldots$ be orthogonal with respect to w, where $\psi_k \in U_k \backslash \{0\}$, $k \geqslant 1$. Let $\{\xi_j^n\}_{j=1}^n$ be the zeroes of ψ_{n+1}, $n \geqslant 1$. Let $f \in C(I)$. If $j \in \{1, 2\}$ and $\theta \in \{a, b\}$ is any zero of c_j we will assume that the one-sided limit* $\lim_{x \to \theta}(f(x) + (-1)^{j-1} \check{f}(x))/c_j(x)$

exists and is finite. For each $n \in \mathbb{N}$, *let* $\chi_n \in V_n$ *interpolate to* f *at* $\{\xi_j^n\}_{j=1}^n$. *Then*

$$\lim_{n\to\infty} \|f-\chi_n\| = 0. \tag{3.5}$$

Proof. By Proposition 3.3 there are exactly n zeroes of ψ_{n+1}. Since they are contained in (a, b) and are situated symmetrically around μ, χ_n exists by Lemma 3.1, $n \geqslant 1$. By (1.3) we may write χ_n in the form $\chi_n = c_1(p_n \circ s) + c_2(q_n \circ s)$, $p_n, q_n \in P_n$, p_n even, q_n odd. It is sufficient to show that $c_1(p_n \circ s)$ converges to the even part of f, $f_1 = \frac{1}{2}(f + \check{f})$, and that $c_2(q_n \circ s)$ converges to the odd part, $f_2 = \frac{1}{2}(f - \check{f})$. By symmetry it is sufficient to show the first. By the hypotheses we may define f_1/c_1 as an element of $C(I)$. Put $v(y) = w(s^{-1}(y))(s^{-1})'(y)$. We notice that

$$\begin{aligned}\|f_1 - c_1(p_n \circ s)\|^2 &= \int_{s(a)}^{s(b)} c_1^2(s^{-1}(y)) \left(\frac{f_1(s^{-1}(y))}{c_1(s^{-1}(y))} - p_n(y)\right)^2 v(y)\, dy \\ &\leqslant \|c_1\|_\infty^2 \int_{s(a)}^{s(b)} \left(\frac{f_1(s^{-1}(y))}{c_1(s^{-1}(y))} - p_n(y)\right)^2 v(y)\, dy.\end{aligned} \tag{3.6}$$

By (3.3), p_n interpolates to $f_1 \circ s^{-1}/c_1 \circ s^{-1}$ on $\{s(\xi_j^n)\}_{j=1}^n$. The change of variable $x = s^{-1}(y)$ shows that $\psi_{n+1} \circ s^{-1}$ is the $(n+1)$th orthogonal polynomial with respect to v. Hence the classical Erdös–Turan theorem shows that the right side of (3.6) tends to zero as $n \to \infty$. ∎

If $c_i(x) > 0$, $x \in I$, $i = 1, 2$, then by the Banach–Steinhaus theorem

$$\|f - \chi_n\| \leqslant C \operatorname{dist}_\infty(f, V_n), \qquad n \geqslant 1 \tag{3.7}$$

where C is a constant. The Jackson-type theorems in [5] now yield the following corollary.

COROLLARY 3.5. *Assume* $s, c_1, c_2 \in C^\infty(I)$ *and that* s', c_1, c_2 *are positive everywhere on* I. *Let* w *be an even weight function. Let* $f \in C^k(I)$, *where* $k \in \mathbb{N}$. *Let* χ_n *be as in Theorem* 3.4. *Then*

$$\|f - \chi_n\| = O\left(\left(\frac{b-a}{n}\right)^k\right), \qquad \textit{as} \quad n \to \infty. \tag{3.8}$$

Let us now look at spaces of trigonometric and hyperbolic functions.

DEFINITION 3.6. Let μ be the midpoint of I. First let $0 < \alpha \leqslant \pi/(b-a)$. Set $S_0^\alpha = \{0\}$ and for $n \geqslant 1$, $x \in I$

$$S_n^\alpha = \left\{ \sum_{j=0}^{m} (a_j \cos(2j\alpha(x-\mu)) + b_j \sin(2j\alpha(x-\mu))) \colon a_j, b_j \in \mathbb{R} \right\}, \qquad n = 2m+1$$

$$= \left\{ \sum_{j=1}^{m} (a_j \cos((2j-1)\,\alpha(x-\mu)) + b_j \sin((2j-1)\,\alpha(x-\mu))) \colon a_j, b_j \in \mathbb{R} \right\},$$

$$n = 2m \tag{3.9}$$

$$\begin{aligned} \tilde{S}_n^\alpha = \operatorname{span}(S_{n-1}^\alpha \cup \{\sigma_n^\alpha\}),\ \sigma_n^\alpha(x) &= \cos(n\alpha(x-\mu)), && n \text{ odd} \\ &= \sin(n\alpha(x-\mu)), && n \text{ even} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} S_{n,\nu}^\alpha &= S_n^\alpha, && n-\nu \text{ even}, \\ &= \tilde{S}_n^\alpha, && n-\nu \text{ odd}, \end{aligned} \qquad \nu = 1, 2. \tag{3.11}$$

Now let α be any positive real number. Define H_n^α, $\tilde{H}_n^\alpha$, $H_{n,\nu}^\alpha$ as in (3.9), (3.10), (3.11) respectively, by just changing sin into sinh and cos into cosh.

Let us now verify that these spaces are special cases of (1.1). Consider $S_{n,1}^\alpha$. Put $s(x) = \sin(\alpha(x-\mu))$, $c_1(x) = 1$, $c_2(x) = \cos(\alpha(x-\mu))$. By elementary trigonometric formulas it follows that $S_{n,1}^\alpha$ is in the form (1.1). Similarly $S_{n,2}^\alpha = V_n$ for $s(x) = \sin(\alpha(x-\mu))$, $c_1(x) = \cos(\alpha(x-\mu))$, $c_2(x) = 1$. The hyperbolic cases follow analogously, $H_{n,\nu}^\alpha = V_n$ when $s(x) = \sinh(\alpha(x-\mu))$,

$$\begin{aligned} c_1(x) &= 1, && \nu = 1 \\ &= \cosh(\alpha(x-\mu)), && \nu = 2, \\ c_2(x) &= \cosh(\alpha(x-\mu)), && \nu = 1 \\ &= 1, && \nu = 2. \end{aligned}$$

Notice that S_n^α is invariant under translation but that $\tilde{S}_n^\alpha$ is not, and similarly for H_n^α and $\tilde{H}_n^\alpha$. If $0 < \alpha < \pi/(b-a)$ then S_n^α is an extended Tchebycheff space, but $\tilde{S}_n^\alpha$ need not be one. In [7] a Newton form for Hermite interpolation by $S_{2n+1}^{1/2}$ and by $\tilde{S}_{2n}^{1/2}$ was given, in the last case assuming the average of the interpolation points to be μ. Error formulas based on trigonometric divided differences are given in [6]. By [13, Theorem 3.1], H_n^1 and hence H_n^α is an extended complete Tchebycheff space when $\alpha > 0$.
Let us now apply our previous results to these functions.

THEOREM 3.7. *Let $I = [a, b]$, $\mu = \frac{1}{2}(a+b)$, and $0 < \alpha < \pi/(b-a)$. Then in both cases, $S_{n,1}^\alpha$, $n = 1, 2, 3, \ldots$, and $S_{n,2}^\alpha$, $n = 1, 2, \ldots$, (2.4) is valid, with $s(x) = \sin(\alpha(x-\mu))$. For any $n \in \mathbb{N}$ and any points $\xi_1, \ldots, \xi_n$ symmetrically located* *around μ, there is a unique $\phi \in S^{\alpha}_{n,\nu}$ that solves* (3.2). *Let w be an even weight function. Let $\psi_1, \psi_2, \psi_3,\ldots$ be the orthogonal sequence equivalent to* 1, $\sin(\alpha(x-\mu))$, $\sin^2(\alpha(x-\mu))$, $\sin^3(\alpha(x-\mu)),\ldots$. *Then ψ_{n+1} possesses exactly n zeroes on I, $\{\xi_j^n\}_{j=1}^n$, and they are located symmetrically around μ, $n \geq 1$. Given any $f \in C(I)$, for each $n \in \mathbb{N}$, let $\chi_n \in S^{\alpha}_{n,\nu}$ interpolate to f at $\{\xi_j^n\}_{j=1}^n$. Then* (3.5) *holds. If $f \in C^k(I)$ then* (3.8) *holds. The same results hold for the hyperbolic case. Here α may be any positive number and $\psi_1, \psi_2, \psi_3,\ldots$ the orthogonal sequence equivalent to* 1, $\sinh(\alpha(x-\mu))$, $\sinh^2(\alpha(x-\mu)),\ldots$.

Let us look at the special case $S^{\alpha}_{n,1}$, $n = 1, 2, 3,\ldots$, when $\alpha = \pi/(b-\alpha)$. Equation (2.4) is still valid. If the symmetrically located interpolation points are lying inside (a, b), then by Lemma 3.1 the interpolation problem (3.2) will always have a unique solution. Suppose that $y \to w(\mu + \alpha^{-1} \arcsin y)\, \alpha^{-1}(1-y^2)^{-1/2}$ is Riemann-integrable on $[-1, 1]$, where w is even. Then by Proposition 3.3 the n simple zeroes of ψ_{n+1} are lying symmetrically around μ and are contained in (a, b). To obtain the conclusion (3.5) we must suppose that $\lim_{x \to a+} (f(x) - f(2\mu - x))/\cos(\alpha(x-\mu))$ and $\lim_{x \to b-} (f(x) - f(2\mu - x))/\cos(\alpha(x-\mu))$ are finite. This will be the case if $f(a) = f(b)$ and f is differentiable at a and b.

The case $S^{\alpha}_{n,2}$, $n = 1, 2,\ldots$, $\alpha = \pi/(b-a)$, has nearly all the properties of the previous case. The only difference is that instead of $f(a) = f(b)$ we suppose that $f(a) = -f(b)$ to obtain (3.5).

In the polynomial case we know that the zeroes of the Tchebycheff polynomials are especially well suited as interpolation points. Consider the proof of Theorem 3.4. If we let v be the Tchebycheff weight function then p_n will normally be a good approximation to $f_1 \circ s^{-1}/c_1 \circ s^{-1}$ in sup-norm and similarly for q_n. Hence $\chi_n = c_1(p_n \circ s) + c_2(q_n \circ s)$ may approximate f well. If s is normalized so that $s(I) = [-1, 1]$, then the interpolation points will be

$$\xi_j = s^{-1}\left(\cos\left(\frac{2j-1}{2n}\pi\right)\right), \qquad j = 1,\ldots, n. \tag{3.12}$$

Particularly, for the trigonometric case,

$$\xi_j = \alpha^{-1} \arcsin\left(\sin\left(\frac{\alpha h}{2}\right)\cos\left(\frac{2j-1}{2n}\pi\right)\right) + \mu, \qquad j = 1,\ldots, n \tag{3.13}$$

where $h = b - a$. As $h \to 0$ or $\alpha \to 0$ these tend to the Tchebycheff points in I. When $\alpha h = \pi$ they are equidistributed.

4. Pointwise and Uniform Convergence of Orthogonal Expansions

We now investigate conditions for the convergence of the Fourier series

$$\sum_{k=1}^{\infty} (f, \bar{\phi}_k)\, \bar{\phi}_k(x) \tag{4.1}$$

to $f(x)$, where $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \ldots$ are the orthonormal functions from V_1, V_2, $V_3, \ldots$, respectively, and the inner product is given by (2.1). We will show that the kernel

$$K_n(x, y) = \sum_{k=1}^{n} \bar{\phi}_k(x)\, \bar{\phi}_k(y), \qquad n \geqslant 1 \tag{4.2}$$

has a Christoffel–Darboux type representation. This leads to Dini-like conditions for the pointwise convergence of (4.1) to f.

Let $\Phi_n f$ be the nth partial sum of (4.1),

$$\Phi_n f = \sum_{k=1}^{n} (f, \bar{\phi}_k)\, \bar{\phi}_k, \qquad n \geqslant 1. \tag{4.3}$$

Φ_n is the orthogonal projection onto V_n. Also

$$(\Phi_n f)(x) = (K_n(x, \cdot), f), \qquad n \geqslant 1. \tag{4.4}$$

The expansion for $K_n(x, y)$ may be simplified.

Proposition 4.1. *Let K_n be as in* (4.2). *Set*

$$\kappa_{i,j}(x, y) = \bar{\phi}_j(x)\, \bar{\phi}_i(y) - \bar{\phi}_i(x)\, \bar{\phi}_j(y), \qquad x, y \in I,\ i, j \geqslant 1. \tag{4.5}$$

If $s^2(x) \neq s^2(y)$ then for $n \geqslant 1$

$$K_n(x, y) = \frac{(\beta_{n+1,n+2}\kappa_{n,n+2} + \beta_{n+1,n+1}\kappa_{n,n+1} + \beta_{n,n+1}\kappa_{n-1,n+1})(x, y)}{s^2(x) - s^2(y)}. \tag{4.6}$$

Proof. The proof is quite similar to that of the ordinary Christoffel–Darboux identity; e.g., [1, pp. 118–119]. Let $n = k+1$ in (2.5). We evaluate it at x, multiply the equation by $\bar{\phi}_k(y)$, switch x and y, and subtract. Then we sum up to n. Because $\beta_{i,j} = \beta_{j+1,i-1}$, nearly all members cancel, and we end up with (4.6). ∎

We now come to the Dini-like conditions for the pointwise convergence to f. This extends the classical case (e.g., [9, pp. 69, 70]).

THEOREM 4.2. *Suppose that $s, c_1, c_2 \in C^2(I)$ satisfy $s'(x)$, $c_1(x)$, $c_2(x) > 0$, $x \in I$. Let w be a weight function. Given any $f \in L^2_w(I)$ and $x \in I \backslash \{\mu\}$, set*

$$F_x(y) = \frac{f(y) - f(x)}{s(y) - s(x)}, \qquad y \in I. \tag{4.7}$$

If F_x, $F_{2\mu - x} \in L^2_w(I)$ and $\{\bar{\phi}_k(x)\}_{k=1}^{\infty}$ is a bounded sequence, then (4.1) converges to $f(x)$ at the point x. The same conclusion is obtained if F_x, $F_{2\mu - x} \in L^1_w(I)$ and $\{\|\bar{\phi}_k\|_\infty\}_{k=1}^{\infty}$ is bounded.

Suppose now that $f \in C(I)$. The Fourier series (4.1) converges to f uniformly on a compact set E if E lies symmetrically around μ, $\mu \notin E$, $\{\sup_{x \in E} |\bar{\phi}_k(x)|\}_{k=1}^{\infty}$ is bounded, and for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_{\max(a, x - \delta)}^{\min(b, x + \delta)} F_x^2(y)\, w(y)\, dy < \varepsilon, \qquad x \in E. \tag{4.8}$$

Proof. Let $f \in L^2_w(I)$. We will instead use

$$g(y) = f(y) + \frac{f(2\mu - x) - f(x)}{2s(x)\, c_2(x)} \phi_2(y), \qquad y \in I.$$

g satisfies $g(2\mu - x) = g(x)$. Since $\phi_2 \in V_n$, $n \geqslant 2$, $\lim_{n \to \infty} (\Phi_n f)(x) = f(x) \Leftrightarrow \lim_{n \to \infty} (\Phi_n g)(x) = g(x)$. Let $i_n \in V_n$ be the best approximation to the function 1 in sup-norm. Since $\Phi_n i_n = i_n$, we may split the problem in three parts:

$$\begin{aligned} g(x) - (\Phi_n g)(x) &= g(x)(1 - i_n(x)) + g(x)(\Phi_n(i_n - 1))(x) \\ &\quad + (\Phi_n(g(x) - g))(x). \end{aligned} \tag{4.9}$$

Suppose now that F_x, $F_{2\mu - x} \in L^2_w(I)$ and that $\{\bar{\phi}_k(x)\}_{k=1}^{\infty}$ is bounded. We will show that each of the three addends on the right side of (4.9) tends to zero as $n \to \infty$.

Since the function 1 is in C^2, then by the Jackson estimate (4.17) in [5], $\|i_n - 1\|_\infty \leqslant Mn^{-2}$, $n > 2$, where M is a constant. So the first addend on the right side of (4.9) is $O(n^{-2})$ and hence tends to 0 as $n \to \infty$.

Consider $|(\Phi_n(i_n - 1))(x)|$. By (4.2), (4.4), and the above estimate for $\|i_n - 1\|_\infty$, this is $\leqslant \sum_{k=1}^{n} |\bar{\phi}_k(x)| \int_a^b |\bar{\phi}_k(y)|\, w(y)\, dy \cdot Mn^{-2}$. By the Cauchy–Schwartz inequality each of these integrals is $\leqslant (\int_a^b w(y)\, dy)^{1/2}$. Since $\{|\bar{\phi}_k(x)|\}_{k=1}^{\infty}$ is bounded, $|(\phi_n(i_n - 1))(x)| = O(n^{-1})$.

Now consider the third term on the right side of (4.9). We may now proceed as in [9, pp. 68, 69]. The third term is

$$\begin{aligned} &\beta_{n+1,n+2}(\kappa_{n,n+2}(x, \cdot), G_x) + \beta_{n+1,n+1}(\kappa_{n,n+1}(x, \cdot), G_x) \\ &\quad + \beta_{n,n+1}(\kappa_{n-1,n+1}(x, \cdot), G_x) \end{aligned} \tag{4.10}$$

where $G_x(y) = (g(x) - g(y))/(s^2(x) - s^2(y))$. By $g(x) = g(2\mu - x)$, $s(2\mu - x) = -s(x)$ and some calculation, we find

$$G_x(y) = \frac{1}{2s(x)} (F_x(y) - F_{2\mu - x}(y)) \qquad (4.11)$$
$$+ \frac{f(2\mu - x) - f(x)}{4s^2(x)\, c_2(x)} \left(\frac{\phi_2(x) - \phi_2(y)}{s(x) - s(y)} - \frac{\phi_2(2\mu - x) - \phi_2(y)}{s(2\mu - x) - s(y)} \right).$$

The second addend on the right side of (4.11) is bounded. The first addend is in $L_w^2(I)$. Hence $G_x \in L_w^2(I)$. Therefore $\lim_{k \to \infty} (\bar{\phi}_k, G_x) = 0$. From (2.5) it follows that $|\beta_{i,j}| \leqslant \|s\|_\infty^2$. The form of $\kappa_{i,j}$ and the boundedness of $\{|\bar{\phi}_k(x)|\}_{k=1}^\infty$ imply that the three inner products in (4.10) tend to zero as $n \to \infty$. Hence the series (4.1) converges to $f(x)$.

Suppose instead that $F_x, F_{2\mu - x} \in L_w^1(I)$ and that $\{\|\bar{\phi}_k\|_\infty\}_{k=1}^\infty$ is bounded. By [9, proof of Theorem 3, pp. 69, 70], $(F_x, \bar{\phi}_k)$, $(F_{2\mu - x}, \bar{\phi}_k) \to 0$ as $k \to \infty$. And that suffices.

Suppose now that $f \in C(I)$, $\mu \notin E$, E compact and symmetric, $\{\sup_{x \in E} |\bar{\phi}_k(x)|\}_{k=1}^\infty$ bounded, and that the uniform Dini-condition (4.8) holds. To show uniform convergence it is enough to look at the third term on the right side of (4.9), i.e., (4.10). By the boundedness of $|\beta_{i,j}|\, |\bar{\phi}_k(x)|$, $x \in E$, $i, j, k \geqslant 1$, it suffices to show that $\lim_{k \to \infty} (\bar{\phi}_k, G_x) = 0$ uniformly for $x \in E$. The second addend on the right side of (4.11) is continuous and hence satisfies (4.8). We may therefore concentrate on F_x, $F_{2\mu - x}$. We need only have $\lim_{n \to \infty} (\bar{\phi}_n, F_x) = 0$ uniformly on E. Since the limit holds pointwise it is sufficient to show that $\{F_x\}_{x \in E}$ is a compact subset of $L_w^2(I)$. Take an arbitrary sequence $\{F_{x_n}\}_{n=1}^\infty$. Proceeding to a subsequence if necessary, we may suppose $\lim_{n \to \infty} x_n = x' \in E$. By (4.8) there is an open neighborhood B of x' such that $\|F_{x'} - F_{x_n}\|_{L_w^2(B)} < \varepsilon$, $n \geqslant n_0$. On $I \setminus B$ $|F_{x'} - F_{x_n}|$ is uniformly bounded for n sufficiently high and $|F_{x'} - F_{x_n}|$ converges pointwise to 0. By Lebesgue's theorem, $F_{x_n} \to F_{x'}$ in $L_w^2(I \setminus B)$. Therefore F_{x_n} converges to $F_{x'}$ in $L_w^2(I)$. So $\{F_x\}_{x \in E}$ is compact and the theorem is proved. ∎

Consider the hypotheses of the last theorem. When is $F_x, F_{2\mu - x} \in L_w^2(I)$? Certainly if f is Lipschitz-continuous. If w is bounded, then F_x, $F_{2\mu - x} \in L_w^2(I)$ when $f \in \mathrm{Lip}\, \alpha$ and $\frac{1}{2} < \alpha \leqslant 1$. When is $\{\bar{\phi}_k(x)\}_{k=1}^\infty$ bounded? We will only consider the case $w = 1$. Proceeding similarly to [4, pp. 27, 28] we find that if s^{-1}, c_1, c_2 are analytic, and $s(I) = [-1, 1]$, then $|\bar{\phi}_n(x)| \leqslant C(1 - s^2(x))^{-1/2}$, $a < x < b$, $n \geqslant 1$, where C is a constant. Hence in the case $w = 1$ the Fourier series converges to $f(x)$ at the point x if $f \in \mathrm{Lip}\, \alpha$, $\frac{1}{2} < \alpha \leqslant 1$, and $a < x < b$, $x \neq \mu$.

Note that the conditions on s, c_1, c_2 in Theorem 4.2 certainly are fulfilled in the trigonometric case (when $0<\alpha<\pi/(b-a)$) and in the hyperbolic case ($\alpha>0$). We also have that s^{-1}, c_1, c_2 are analytic.

Consider the proof of Theorem 4.2. Suppose that $c_1=1$. Then of course $i_n=1$. So (4.9) consists of only its third term. We may then weaken the conditions on s, c_1, c_2. Instead of $s'(y)$, $c_2(y)>0$ $\forall y\in I$ we may suppose that Assumption 3.2 is fulfilled and that $a<x<b$, $x\neq\mu$. For the uniform convergence part we need to assume that $E\subset(a,b)$, $\mu\notin E$. Notice that this covers the case $V_n=S^{\alpha}_{n,1}$, $n=1,2,3,\ldots$, $\alpha=\pi/(b-a)$. From the classical theory of Fourier series we know that $F_x\in L^1_w(I)$ is sufficient to get convergence towards $f(x)$ at x. Here we had to assume $F_{2\mu-x}\in L^1_w(I)$ and $x\notin\{a,\mu,b\}$ too, to get the same result. But then the series would also converge at $2\mu-x$.

5. Extension of Gaussian Quadrature

It has been noted in [10] that there exist trigonometric and exponential analogs of Gaussian quadrature formulas. In [12] this also was observed for the trigonometric case. Here we construct n-point rules that are exact for the more general class V_{2n}, $n\geqslant 1$. We will find upper bounds for the quadrature error. Also exact formulas for the quadrature error are furnished in the trigonometric and hyperbolic case. For small intervals we find a way to make a comparison with ordinary Gaussian quadrature. Some numerical examples are given.

But first we must show the existence of such formulas.

Theorem 5.1. *Let w be a positive, even, and Riemann-integrable weight function on $I=[a,b]$. Suppose that Assumption 3.2 is fulfilled. Let $\{\psi_k\}_{k=1}^{\infty}$, $\psi_k\in U_k=\operatorname{span}\{1,s,s^2,\ldots,s^{k-1}\}$, be the orthogonal sequence generated by the weight function $\tilde{w}=c_1w$, and normalized so that $\psi_k-s^{k-1}\in U_{k-1}$, $k\geqslant 1$. For each $n\geqslant 1$ we then have*

$$\int_a^b \phi(x)\,w(x)\,dx=\sum_{j=1}^{n}A_j\phi(\xi_j),\qquad \phi\in V_{2n}\tag{5.1}$$

where $\{\xi_j\}_{j=1}^n$ are the zeroes of ψ_{n+1} and

$$A_j=\frac{\|\psi_n\|^2_{\tilde{w}}}{\psi'_{n+1}(\xi_j)\,\psi_n(\xi_j)}\cdot\frac{s'(\xi_j)}{c_1(\xi_j)}>0,\qquad j=1,\ldots,n\tag{5.2}$$

and where $\|\ \|_{\tilde{w}}$ denotes the norm induced by $\tilde{w}$.

Proof. By linearity it is sufficient to show (5.1) for $\phi = \phi_k$, $k = 1,..., 2n$. Since w is even, (2.3), (1.4), and (1.5) imply

$$\phi_k = (p_k \circ s) \begin{cases} c_1, & k \text{ odd} \\ c_2, & k \text{ even} \end{cases}$$

where $p_k \in \mathbb{P}_k$ is monic and p_k is even (odd) if k is odd (even). First take the case k odd. The change of variable $x = s^{-1}(y)$ gives

$$\int_a^b \phi_k(x)\, w(x)\, dx = \int_J p_k(y)\, \tilde{v}(y)\, dy,$$

where $J = s(I)$ and (5.3)

$$\tilde{v}(y) = \tilde{w}(s^{-1}(y))(s^{-1})'(y), \qquad y \in J.$$

The two integrals in (5.3) are also equal if k is even because then they both are equal to zero.

Let $\{q_j\}_{j=1}^{\infty}$ be the monic polynomials that are orthogonal with respect to the weight function $\tilde{v}$, where $q_j \in P_j$, $j \geqslant 1$. Let $(\ , \ )_{\tilde{v}}$ be the inner product and $\| \ \|_{\tilde{v}}$ be the norm. Let $\{\rho_j\}_{j=1}^{n}$ be the zeroes of q_{n+1}, ordered increasingly, and put

$$B_j = \|q_n\|_{\tilde{v}}^2 / \{q'_{n+1}(\rho_j)\, q_n(\rho_j)\}, \qquad j = 1,..., n. \tag{5.4}$$

By the theory for ordinary Gaussian quadrature (e.g., [2])

$$\int_J p_k(y)\, \tilde{v}(y)\, dy = \sum_{j=1}^{n} B_j\, p_k(\rho_j), \qquad k = 1,..., 2n. \tag{5.5}$$

Put $\xi_j = s^{-1}(\rho_j)$, $A_j = B_j / c_1(\xi_j)$, $j = 1,..., n$. $\tilde{v}$ even implies $\rho_{n+1-j} = -\rho_j$, $B_{n+1-j} = B_j$, $j = 1,..., n$. Because s is odd and c_1 is even, $\xi_{n+1-j} = 2\mu - \xi_j$, $A_{n+1-j} = A_j$, $j = 1,..., n$. First, let k be even. Then ϕ_k is odd. Both $\int_a^b \phi_k(x)\, w(x)\, dx$ and $\sum_1^n A_j \phi_k(\xi_j)$ are zero. Now let k be odd. By (5.3) and (5.5)

$$\int_a^b \phi_k(x)\, w(x)\, dx = \sum_{j=1}^{n} A_j c_1(\xi_j)\, p_k(s(\xi_j)) = \sum_{j=1}^{n} A_j \phi_k(\xi_j).$$

$q_k \circ s$ is a monic element of U_k. Also, for $i \neq j$,

$$(q_i \circ s, q_j \circ s)_{\tilde{w}} = (q_i, q_j)_{\tilde{v}} = 0.$$

Therefore $\psi_k = q_k \circ s$. Hence $\{\xi_j\}_{j=1}^{n}$ are the zeroes of ψ_{n+1}. Equation (5.2) follows from (5.4) by noting that $\|q_n\|_{\tilde{v}}^2 = \|\psi_n\|_{\tilde{w}}^2$, $q_n(\rho_j) = \psi_n(\xi_j)$, and $q'_{n+1}(\rho_j) = \psi'_{n+1}(\xi_j)/s'(\xi_j)$. ∎

If f is a given function we will approximate $\int_a^b f(x)\, w(x)\, dx$ by $\sum_{j=1}^n A_j f(\xi_j)$.

Of course, (5.1) holds in the cases $V_{2n} = S^{\alpha}_{2n,\nu}$, $0 < \alpha < \pi/(b-a)$, and $V_{2n} = H^{\alpha}_{2n,\nu}$, $\alpha > 0$, $\nu = 1, 2$, $n \in \mathbb{N}$. If we assume that $w(\mu + \alpha^{-1} \arcsin y)\, \alpha^{-1}(1-y^2)^{-1/2}$ is integrable, then it also holds in the case $V_{2n} = S^{\alpha}_{2n,\nu}$, $\alpha = \pi/(b-a)$, $\nu = 1, 2$.

Consider the case $w = 1$, $V_{2n} = S^{\alpha}_{2n,1}$. Then

$$\xi_j = \mu + \alpha^{-1} \arcsin\left(\sin\left(\frac{\alpha(b-a)}{2}\right)\rho_j\right), \qquad j = 1,..., n$$

where $\rho_1,..., \rho_n$ are the zeroes of $q_{n+1} \in P_{n+1}$, and q_{n+1} is orthogonal to P_n with respect to the weight function

$$\left(1 - \sin^2\left(\frac{\alpha(b-a)}{2}\right)\cdot y^2\right)^{-1/2} \qquad y \in [-1, 1].$$

For small α, $\rho_1,..., \rho_n$ are asymptotically the zeroes of the nth degree Legendre polynomial on $[-1, 1]$. In that case $\xi_1,..., \xi_n$ are nearly the zeroes of the nth degree Legendre polynomial on I. If $\alpha = \pi/(b-a)$, $\rho_1,..., \rho_n$ are the zeroes of the nth degree Tchebycheff polynomial. In that case $\xi_1,..., \xi_n$ are equidistributed and we obtain the well-known trapezoidal rule.

The proof of Theorem 5.1 gives a hint for the practical construction of n-point rules (cf. [3, p. 580]). Let $\kappa_m = \int_{-1}^1 y^{2m}\tilde{v}(y)\, dy$, $m \geq 0$. In both the trigonometric and hyperbolic cases κ_m is found easily when $w = 1$. Set $q_k(y) = \sum_{0 \leq 2j \leq k-1} c_{k,j}\, y^{k-1-2j}$, where $c_{k,0} = 1$. The coefficients $c_{k,j}$ are found via the recurrence relation for orthogonal polynomials

$$c_{k+1,j} = c_{k,j} - b_k c_{k-1,j-1}, \qquad 0 \leq 2j \leq k,$$

where

$$b_k = \sum_{0 \leq 2j \leq k-1} c_{k,j} \kappa_{k-1-j} \Big/ \sum_{0 \leq 2j \leq k-2} c_{k-1,j} \kappa_{k-2-j}.$$

Having found the coefficients in q_{n+1}, we use Newton's method to calculate $\rho_1,..., \rho_n$. Then we find

$$B_j = \sum_{0 \leq 2j \leq n-1} c_{n,j} \kappa_{n-1-j} / (q'_{n+1}(\rho_j)\, q_n(\rho_j)), \qquad j = 1,..., n.$$

Finally we put $\xi_j = s^{-1}(\rho_j)$, $A_j = B_j / c_1(\xi_j)$, $j = 1,..., n$.

The quadrature error is defined by

$$E_n(f) = \int_a^b f(x)\, w(x)\, dx - \sum_{j=1}^n A_j f(\xi_j). \tag{5.6}$$

We assume that f is bounded and that fw is integrable.

Since $c_1 = f_1 \in V_{2n}$, $n \geqslant 1$, $0 = E_n(c_1) = \int_a^b c_1(x)\, w(x)\, dx - \sum_{k=1}^n A_k c_1(\xi_k)$. It follows that

$$\sum_{k=1}^{n} |A_k| \leqslant (\max_I c_1 / \min_I c_1) \int_a^b w(x)\, dx. \tag{5.7}$$

The upper bound in (5.7) is independent of n. Hence the method is stable.

Let us look at some numerical examples with $w = 1$. For notational convenience we denote the quadrature error by $E_n^{\mathrm{I}}(f)$ if $V_{2n} = P_{2n}$, by $E_{n,v}^{\mathrm{II}}(f)$ if $V_{2n} = S_{2n,v}^{1/2}$, and by $E_{n,v}^{\mathrm{III}}(f)$ if $V_{2n} = H_{2n,v}^{1/2}$.

EXAMPLE 5.2. Let $f_\lambda(x) = e^{\lambda \cos x}$, $x \in I = [-3, 3]$. We found

n	$E_n^{\mathrm{I}}(f_1)$	$E_{n,1}^{\mathrm{II}}(f_1)$	$E_n^{\mathrm{I}}(f_{0.1})$	$E_{n,1}^{\mathrm{II}}(f_{0.1})$	$E_n^{\mathrm{I}}(f_{0.01})$	$E_{n,1}^{\mathrm{II}}(f_{0.01})$
2	3	2	$1 \cdot 10^{-1}$	$1 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$1 \cdot 10^{-4}$
4	$4 \cdot 10^{-1}$	$3 \cdot 10^{-2}$	$2 \cdot 10^{-3}$	$3 \cdot 10^{-6}$	$6 \cdot 10^{-5}$	$3 \cdot 10^{-10}$
6	$3 \cdot 10^{-2}$	$3 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-7}$	$-3 \cdot 10^{-16}$

EXAMPLE 5.3. Here $f(x) = \ln(2 + x)$, $x \in I = [-0.25, 0.25]$. We found

n	$E_n^{\mathrm{I}}(f)$	$E_{n,1}^{\mathrm{II}}(f)$	$E_{n,2}^{\mathrm{II}}(f)$	$E_{n,1}^{\mathrm{III}}(f)$	$E_{n,2}^{\mathrm{III}}(f)$
2	$-3 \cdot 10^{-6}$	$-5 \cdot 10^{-6}$	$-4 \cdot 10^{-6}$	$-9 \cdot 10^{-7}$	$5 \cdot 10^{-6}$
4	$-2 \cdot 10^{-11}$	$-8 \cdot 10^{-11}$	$-1 \cdot 10^{-10}$	$-3 \cdot 10^{-12}$	$6 \cdot 10^{-11}$
6	$-2 \cdot 10^{-16}$	$-2 \cdot 10^{-15}$	$-4 \cdot 10^{-15}$	$-3 \cdot 10^{-17}$	$8 \cdot 10^{-16}$

In Example 5.3 the errors for the five methods are of about the same size. But in Example 5.2 the trigonometric method works remarkably better. We will later return with an explanation of this phenomenon.

Let us find upper bounds for $|E_n(f)|$.

PROPOSITION 5.4. *Let w be positive, even and Riemann-integrable on $I = [a, b]$. Suppose that Assumption* 3.2 *holds and that $c_1(a) > 0$, $c_1(b) > 0$. If f is bounded and Riemann-integrable, then for $n \geqslant 1$*

$$|E_n(f)| \leqslant M \operatorname{dist}_\infty(f, V_{2n}),$$

where (5.8)

$$M = \left(1 + \frac{\max c_1}{\min c_1}\right) \int_a^b w(x)\, dx.$$

Let $k \in \mathbb{N}$. Suppose further that $s, c_1, c_2 \in C^k(I)$ and that $s'(a)$, $s'(b)$, $c_2(a)$, $c_2(b) > 0$. If $f \in C^k(I)$, then

$$|E_n(f)| = o\left(\left(\frac{b-a}{n}\right)^k\right), \qquad 2n > k. \tag{5.9}$$

Proof. By (5.7) we see that $\|E_n\|_\infty \leqslant M$. Linearity and $E_n(\phi) = 0$ for every ϕ in V_{2n} imply (5.8). The Jackson-type theorems in [5] give (5.9). ∎

If $f \in C(I)$ then by Proposition 5.4, $\lim_{n \to \infty} E_n(f) = 0$. By [2, pp. 100–102] this result may be extended.

PROPOSITION 5.5. *Let w be positive, even, and Riemann-integrable. Suppose that Assumption 3.2 holds and that $c_1(x)$, $c_2(x)$, $s'(x) > 0$, $x = a, b$. If f is bounded and Riemann-integrable, then*

$$\lim_{n \to \infty} E_n(f) = 0. \tag{5.10}$$

If only boundedness and integrability are assumed, then the convergence may be very slow.

For smooth integrands we obtain explicit error formulas in the trigonometric and hyperbolic cases.

THEOREM 5.6. *Let w be a positive, even, and Riemann-integrable weight function on $I = [a, b]$, and let $n \in \mathbb{N}$. First let $V_{2n} = S^\alpha_{2n,v}$, where $0 < \alpha < \pi/(b-a)$. Set $l(x) = \mu + 2\alpha(x - \mu)$ and $\zeta_j = l(\xi_j)$, $j = 1, \ldots, n$. If $f \in C^2(I)$ there is a $\xi \in I$ such that*

$$E_n(f) = D_n(f, \xi) \int_a^b \left(\prod_{j=1}^n \sin(\alpha(x - \xi_j))\right)^2 w(x)\, dx, \tag{5.11}$$

where

$$\begin{aligned} D_n(f, x) &= \lambda[\zeta_1, \zeta_1, \ldots, \zeta_n, \zeta_n, l(x)]_t(f \circ l^{-1}), \qquad v = 1 \\ &= [\zeta_1, \zeta_1, \ldots, \zeta_n, \zeta_n, l(x)]_t(f \circ l^{-1}), \qquad v = 2 \end{aligned}$$

and where $\lambda[\ ,\ ,\ ]_t F$, $[\ ,\ ,\ ]_t F$ are trigonometric divided differences (see [7]).

If $f \in C^{2n}(I)$ then there is a $\bar{\xi} \in I$ such that

$$E_n(f) = c(\xi) \int_a^b \left(\prod_{j=1}^n \alpha^{-1} \sin(\alpha(x - \xi_j))\right)^2 w(x)\, dx \cdot L^\alpha_{2n,v} f(\bar{\xi})/(2n)!, \tag{5.12}$$

where

$$L^{\alpha}_{2n,\nu}f(y)=\left(\left(\cos\frac{l(\xi)-y}{2}D\right.\right.$$

$$\left.\left.+2\alpha n\sin\frac{l(\xi)-y}{2}\right)D\prod_{j=1}^{n-1}(D^2+(2\alpha j)^2)\right)f(y), \qquad \nu=1$$

$$=\prod_{j=1}^{n}\left(D^2+\left(2\alpha\left(j-\frac{1}{2}\right)\right)^2\right)f(y), \qquad \nu=2$$

and $D=d/dy$, $c(\xi)=n\int_a^b T_{0,2n}(x)\,dx$, *where* $T_{0,2n}$ *is the trigonometric B-spline with knots* $\zeta_1, \zeta_1,..., \zeta_n, \zeta_n, l(\xi)$.

Now let $V_{2n}=H^{\alpha}_{2n,2}$, *where* $\alpha>0$. *Set* $l(x)=\mu+\alpha(x-\mu)$, $\zeta_j=l(\xi_j)$, $j=1,..., n$. *Put* $D_n(f, x)=[\zeta_1, \zeta_1,..., \zeta_n, \zeta_n, l(x)]_h(f\circ l^{-1})$, *where* $[\ ,\ ,\ ]_h F$ *is the hyperbolic divided difference (see* [13, (4.1)]). *Then if* $f\in C^2(I)$ *there is a* $\xi\in I$ *such that*

$$E_n(f)=D_n(f, \xi)\int_a^b\left(\prod_{j=1}^{n}\sinh(\alpha(x-\xi_j))\right)^2 w(x)\,dx. \qquad (5.13)$$

Let $c(\xi)=2n\int_a^b Q_{0,2n}(x)\,dx$, *where* $Q_{0,2n}$ *is the hyperbolic B-spline with knots* $\zeta_1, \zeta_1,..., \zeta_n, \zeta_n, l(\xi)$ *(see* [13, (5.1)]). *Set* $L^{\alpha}_{2n,2}=\prod_{j=1}^{n}(D^2-(2\alpha(j-\frac{1}{2}))^2)$. *If* $f\in C^{2n}(I)$ *there is a* $\bar{\xi}\in I$ *such that*

$$E_n(f)=c(\xi)\int_a^b\left(\prod_{j=1}^{n}\alpha^{-1}\sinh(\alpha(x-\xi_j))\right)w(x)\,dx\cdot L^{\alpha}_{2n,2}f(\bar{\xi})/(2n)!. \qquad (5.14)$$

Proof. The proof is analogous to that in the polynomial case. Take the trigonometric cases first. We seek an element Q of $S^{\alpha}_{2n,\nu}$ that solves the Hermite interpolation problem

$$Q(\xi_j)=f(\xi_j), \qquad Q'(\xi_j)=f'(\xi_j), \qquad j=1,..., n.$$

We simply let $Q=R\circ l$, where $R\in S^{1/2}_{2n,\nu}$ Hermite interpolates to $F=f\circ l^{-1}$ at the ζ's. By Theorem 5.1

$$\sum_{j=1}^{n}A_jf(\xi_j)=\sum_{j=1}^{n}A_jQ(\xi_j)=\int_a^b Q(x)\,w(x)\,dx$$

so $E_n(f)=\int_a^b(f(x)-Q(x))\,w(x)\,dx=(1/2\alpha)\int_{\mu-\alpha h}^{\mu+\alpha h}(F(y)-R(y))\,w(l^{-1}(y))\,dy$ where $h=b-a$. By [6, (3.1), (4.1)]

$$F(y)-R(y)=\left(\prod_{j=1}^{n}\sin\frac{y-\zeta_j}{2}\right)^2\begin{cases}\lambda[\zeta_1, \zeta_1,..., \zeta_n, \zeta_n, y]_t F, & \nu=1\\ [\zeta_1, \zeta_1,..., \zeta_n, \zeta_n, y]_t F, & \nu=2.\end{cases}$$

Since $F \in C^2(I)$ the trigonometric divided differences are continuous in y. A change of variable plus the mean value theorem for integrals now gives (5.11). The integral representation for trigonometric divided differences [8, p. 276; 6, (4.3)] implies (5.12).

The hyperbolic case, $V_{2n} = H^{\alpha}_{2n,2} = H^{\alpha}_{2n}$, is proved in the same way. After having redefined l we set $F = f \circ l^{-1}$ and let $R \in H^1_{2n}$ solve the Hermite interpolation problem at the ζ_j's. Put $Q = R \circ l$ and proceed as above. We get $E_n(f) = (1/\alpha) \int_{\mu - \alpha h/2}^{\mu + \alpha h/2} (F(y) - R(y))\, w(l^{-1}(y))\, dy$. Using the representation (4.2) in [13] for the hyperbolic divided difference and using the smoothness properties of these, we obtain

$$F(y) - R(y) = \left(\prod_{j=1}^{n} \sinh(y - \zeta_j) \right)^2 [\zeta_1, \zeta_1, ..., \zeta_n, \zeta_n, y]_h F.$$

This shows (5.13). Equation (5.14) follows from the integral representation of hyperbolic divided differences [13, (7.5)]. ∎

But how should one choose between these methods? For smooth integrands on short intervals we will now see that this depends on the action of the corresponding differential operator upon the integrand.

THEOREM 5.7. *Let v_0 be positive, even, and Riemann-integrable on $[-1, 1]$. Set $w(x) = v_0((2/h)(x - \mu))$, $x \in I_h = [\mu - h/2, \mu + h/2]$. Let $n \in \mathbb{N}$. Suppose that $f \in C^{2n}(I_{h_0})$ for an $h_0 \in (0, \pi/\alpha)$. Let V_{2n} be $S^{\alpha}_{2n,\nu}$ restricted to I_h. Denote by $E^h_n(f)$ ($E^h_{n,\nu}(f)$) the quadrature error in n-point (trigonometric) Gaussian quadrature. If $D^{2n} f(\mu) \neq 0$ then*

$$\lim_{h \to 0+} \frac{E^h_{n,\nu}(f)}{E^h_n(f)} = \frac{L^{\alpha}_{2n,\nu} f(\mu)}{D^{2n} f(\mu)}, \tag{5.15}$$

where here

$$\begin{aligned} L^{\alpha}_{2n,\nu} &= \prod_{j=0}^{n-1} (D^2 + (2\alpha j)^2), & \nu = 1 \\ &= \prod_{j=1}^{n} \left(D^2 + \left(2\alpha \left(j - \frac{1}{2} \right) \right)^2 \right), & \nu = 2. \end{aligned}$$

The same conclusion holds for the hyperbolic case $V_{2n} = H^{\alpha}_{2n,2}$ if $L^{\alpha}_{2n,2}$ is changed to $\prod_{j=1}^{n} (D^2 - (2\alpha(j - \frac{1}{2}))^2)$.

Proof. Let us take the trigonometric case first. So $s(x) = \sin(\alpha(x - \mu))/\sin(\alpha h/2)$. Let $\{p_{k,0}\}_{k=1}^{\infty}$ be the monic, orthogonal polynomials associated with v_0, and $\{p_{k,h}\}_{k=1}^{\infty}$ the corresponding family with respect to $\tilde{v}_h(y) = c_1(s^{-1}(y)) \cdot (s^{-1})'(y) \cdot v_0((2/h)(s^{-1}(y) - \mu))$. Set $v_h(y) = v_0((2/h)(s^{-1}(y) - \mu))(s^{-1})'(y)$, $|y| \leq 1$. Let the ξ_j's lie in ascending order. By

Proposition 3.3, $\xi_{n+1-j} = 2\mu - \xi_j$, $j = 1, \ldots, n$. By the trigonometric formula $\sin(A+B)\sin(A-B) = (\sin A + \sin B)(\sin A - \sin B)$ we obtain $\sin(\alpha(x-\xi_j))\sin(\alpha(x-\xi_{n+1-j})) = (\sin(\alpha(x-\mu)) - \sin(\alpha(\xi_j-\mu)))(\sin(\alpha(x-\mu)) + \sin(\alpha(\xi_j-\mu)))$, $j = 1, \ldots, n$. Therefore $\prod_{j=1}^n \sin(\alpha(x-\xi_j)) \in U_{n+1}$. Since it has the same zeroes as ψ_{n+1} of Theorem 5.1 we have $\prod_{j=1}^n \sin(\alpha(x-\xi_j)) = (\sin(\alpha h/2))^n \cdot \psi_{n+1}(x)$. A change of variable shows that $\psi_{n+1} \circ s^{-1}$ is orthogonal to P_n with respect to $\tilde{v}_h$. Hence $p_{n+1,h} = \psi_{n+1} \circ s^{-1}$. Substituting $x = s^{-1}(y)$ in (5.12) now yields

$$E_{n,v}^h(f) = c(\xi_v^h)\left(\alpha^{-1}\sin\left(\frac{\alpha h}{2}\right)\right)^{2n}\int_{-1}^1 p_{n+1,h}^2(y)\, v_h(y)\, dy\, L_{2n,v}^\alpha f(\bar{\xi}_v^h)/(2n)!$$

for some $\xi_v^h, \bar{\xi}_v^h \in I_h$.

For the polynomial case

$$E_n^h(f) = \left(\frac{h}{2}\right)^{2n+1}\int_{-1}^1 p_{n+1,0}^2(y)\, v_0(y)\, dy \cdot D^{2n}f(\bar{\xi}^h)/(2n)!$$

for some $\bar{\xi}^h \in I_h$.

As $h \to 0+$, $(\alpha^{-1}\sin(\alpha h/2))^{2n}/(h/2)^{2n} \to 1$. Also $D^{2n}f(\bar{\xi}^h) \to D^{2n}f(\mu)$, $L_{2n,v}^\alpha f(\bar{\xi}_v^h) \to L_{2n,v}^\alpha f(\mu)$, where the last $L_{2n,v}^\alpha$ is that of (5.15). By [6, Proposition 2.1], $c(\xi_v^h) \to 1$ as $h \to 0+$. It remains to show that

$$\frac{2}{h}\int_{-1}^1 p_{n+1,h}^2(y)\, v_h(y)\, dy \to \int_{-1}^1 p_{n+1,0}^2(y)\, v_0(y)\, dy \qquad \text{as} \quad h \to 0+. \tag{5.16}$$

Now

$$\begin{aligned}\frac{2}{h} v_h(y) &= v_0\left(\frac{1}{\alpha h/2}\arcsin\left(\sin\left(\frac{\alpha h}{2}\right) y\right)\right)\left(1 - \sin^2\left(\frac{\alpha h}{2}\right) y^2\right)^{-1/2} \\ &\quad \times \sin\left(\frac{\alpha h}{2}\right)\bigg/\left(\frac{\alpha h}{2}\right) \\ &= v_0((1+O(h^2))\, y + O(h^4))(1+O(h^2)) \to v_0(y)\end{aligned}$$

in L_1-norm as $h \to 0+$. Similarly $(2/h)\,\tilde{v}_h \to v_0$ in L_1-norm. Therefore the coefficients in the recurrence relation for the $p_{k,h}$'s tend to the corresponding coefficients in the recurrence relation for the $p_{k,0}$'s. Hence $\| p_{n+1,h} - p_{n+1,0} \|_\infty \to 0$ as $h \to 0+$. We obtain (5.16).

The conclusion in the hyperbolic case follows in the same manner. $\lim_{h \to 0+} c(\xi_v^h) = 1$ can be shown by the method in [6]. ∎

From the last proof,

$$\frac{E^h_{n,v}(f)}{E^h_n(f)} = \frac{L^\alpha_{2n,v} f(\mu)}{D^{2n} f(\mu)} + O(h) \tag{5.17}$$

if $f \in C^{2n+1}$ in a neighborhood of μ and $D^{2n}f(\mu) \neq 0$.

If $v_0 \equiv 1$ in Theorem 5.7, then by [2, p. 75]

$$\lim_{h \to 0+} h^{-(2n+1)} E^h_{n,v}(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)((2n)!)^3} L^\alpha_{2n,v} f(\mu). \tag{5.18}$$

Let us look back at Example 5.3. Take the simplest case $n = 2$. We find

$$D^4 f(0) = -0.4, \qquad L^{1/2}_{4,1} f(0) = -0.6 = L^{1/2}_{4,2} f(0), \qquad L^{1/2}_{4,2,\text{hyp. case}} f(0) = 0.6.$$

In view of Theorem 5.7 we therefore could have predicted that the quadrature errors for the four methods would be of the same absolute value.

Of course, in most cases it is not so easy to differentiate the integrand. But one can sometimes see if the integrand is of hyperbolic or of trigonometric category. For instance, as we see immediately, the integrands of Example 5.2 are of the trigonometric type.

Let us now analyze why the trigonometric method worked so well in Example 5.2. We are going to be more general. Suppose that $f \in C^{2n+m}(\mathbb{R})$ has period T. Therefore f may be developed in a Fourier series

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(2k\alpha x) + b_k \sin(2k\alpha x)) \tag{5.19}$$

where $\alpha = \pi/T$ and where $|a_k|$, $|b_k| = O(k^{-(2n+m+1)})$, $k \geqslant 1$. Differentiating on both sides of (5.19) yields

$$D^{2n} f(x) = (-1)^n (2\alpha)^{2n} \sum_{k=0}^{\infty} k^{2n} (a_k \cos(2k\alpha x) + b_k \sin(2k\alpha x))$$

$$L^\alpha_{2n,1} f(x) = (-1)^n (2\alpha)^{2n} \sum_{k=0}^{\infty} k^{2n} \Theta_{k,n} (a_k \cos(2k\alpha x) + b_k \sin(2k\alpha x))$$

where $L^\alpha_{2n,1}$ is as in (5.15) and

$$\Theta_{k,n} = \prod_{j=0}^{n-1} \left(1 - \left(\frac{j}{k}\right)^2\right). \tag{5.20}$$

If $I_h = [-h/2, h/2]$ and $D^{2n} f(0) \neq 0$ then by Theorem 5.7

$$\lim_{h \to 0+} \frac{E^h_{n,1}(f)}{E^h_n(f)} = \frac{\sum_{k=n}^{\infty} \Theta_{k,n} k^{2n} a_k}{\sum_{k=1}^{\infty} k^{2n} a_k}. \tag{5.21}$$

Here we have used $\Theta_{k,n} = 0$ for $k < n$. For $k \geqslant n$, $0 < \Theta_{k,n} < e^{-n^3/3k^2}$. So each term in the numerator on the right side of (5.21) is $\leqslant$ the corresponding term in the denominator, in absolute value. We have $|k^{2n}a_k| = O(k^{-m-1})$. Since the numerator lacks terms for $k < n$, the quotient (5.21) is often less than 1 in absolute value.

Of course, the interval in Example 5.2 is not small. But the same tendency holds. As λ decreases, the coefficients in the Fourier series of f_λ decay much more rapidly. The quotient (5.21) decreases quickly.

Hence for some problems it may be an advantage to use these non-polynomial methods.

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